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VIBRATIONAL STATE OF COMPLEX MECHANICAL STRUCTURES UNDER BROAD-BAND EXCITATION†

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Abstract—An integral description of the vibration of weakly coupled complex structures as assemblages of substructures is proposed. The common integral methods are valid only for the frequency domain of the global resonances. Outside this domain (i.e. in the high-frequency region), the vibration localizes within each substructure, which results in considerable spatial absorption of vibration. The effect of the backward influence of the vibration of secondary structures on the general vibrational state of the structure becomes evident. The main distinction between low- and high-frequency vibrations is described. As opposed to the first global resonances which envelop the whole structure, the high-frequency vibration propagates from the source of excitation.

1. INTRODUCTION

Problems regarding the definition of vibrational states have been considered closely by mechanical engineers for many years, because vibration is a standard reaction of any real structure to any external, dynamic load. As remarked in a detailed survey by Noor and Atluri (1987), the development of methods for dynamical simulation of complex structures under broad-band frequency loading (high-energy impact, structural penetration, crash-worthiness, impulsive and stochastic loading, etc.) is one of the urgent questions that should currently attract the attention of the mechanisist. The complexity of analyzing complicated mechanical structures such as buildings, ships, aeroplanes and spacecraft is caused firstly, by the complexity of the observed mechanical structure's shape, then by the collection of separate substructures and finally, by the presence of various equipment fixed on carrier constructions. Besides, the weight of the equipment is comparable to and often exceeds the weight of the carrier structure. It is also evident, as a result of impressive successes of the methods of computational structural dynamics, that the specific weight of internal equipment will increase permanently. At the same time, in the literature of present structural mechanics, the role of the second structure is commonly avoided. It is also necessary to take into account that all the structures in question are weakly coupled. All substructures are weakly coupled to each other, i.e. they are fixed to each other at several points or localized regions only. Besides, the equipment is weakly coupled to the carrier structure, the elements of separate devices are weakly coupled to each other and so on. At the same time we have to keep in mind that each of the elements of the equipment is a complicated mechanical system itself.

Let us suppose that we write down the exact boundary-value problem for the description of the dynamics of the structure, although this presents a problem because of the vagueness of the boundary conditions. Nevertheless, if we assume that the exact solution has been arrived at, the very interpretation of this result is a problem too. The field of vibration of an essentially heterogeneous structure under broad-band loading is an extremely complicated function with respect to time and spatial coordinates, because so many modes are excited in the structure.

If the structure proves to be so complicated that classical methods of vibration theory become unacceptable, it is reasonable to use some integral theories. At first sight it appears to be a paradoxical situation : the more complicated the structure, the simpler its description. But this is merely the first impression. The same situation exists in the theory of thermal conductivity where it proves to be sufficient to introduce the temperature, which is the scalar integral characteristic for describing the three-dimensional, and very complicated, field of molecular velocities.

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2. LOW-FREQUENCY AND HIGH-FREQUENCY VIBRATION OF COMPLICATED MECHANICAL STRUCTURES

Let us consider an actual mechanical structure V, which is formed from a great number of substructures V_n with boundary surfaces S_n ($n = 1, 2, \ldots, N$). The substructures are fixed to one another or to the carrier structure, the latter providing the integrity and rigidity of the entire structure. The sizes of the substructures V_n are much less than that of the structure V. Nevertheless, each substructure is a complex dynamical system.

The standard way of solving this problem is by expansion in terms of the normal modes. Let us seek the solution in a similar way:

$$\mathbf{u}_n(\mathbf{r}, t) = \sum_{k=1}^{\infty} \mathbf{u}_{nk}(\mathbf{r}) q_{nk}(t) + \mathbf{u}_c(\mathbf{r}, t). \tag{1}$$

Here $\mathbf{u}_n(\mathbf{r}, t)$ is the vector of the absolute displacement of the point with coordinate $\mathbf{r} \in V_n$, $\mathbf{u}_{nk}(\mathbf{r})$ is the kth normal mode of substructure V_n, $q_{nk}(t)$ is the generalized coordinate and t is time. The function $\mathbf{u}_c(\mathbf{r}, t)$ is usually introduced in problems of mathematical physics to improve convergence in the vicinity of the boundary (Mindlin and Goodman, 1950). The reason for the addition of $\mathbf{u}_c(\mathbf{r}, t)$ in eqn (1) is, in principle, different. We want an extremely simple boundary problem for $\mathbf{u}_c$, which allows us to produce an engineer's description of the field of vibration of the complex structure. Keeping in mind our desire to create a mathematical description that would be equally suitable for investigating stationary, nonstationary, deterministic and random vibration, let us make use of the following spectral representation:

$$\mathbf{u}_c(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \mathbf{u}_c(\mathbf{r}, \omega) e^{i\omega t} \, d\omega, \tag{2}$$

where hereinafter, the same designation for the spectra will be observed. When beginning to examine the dynamic behaviour of the real system at the whole frequency band, it is necessary to distinguish precisely two frequency regions with essentially different properties. The first region is the low-frequency region. Let us agree to consider the low-frequency region as the frequency domain in which the global resonances of the whole structure appear. This region is not large. Engineering experience convinces us that the first, second and rare third resonances only take place in weakly coupled mechanical systems. The low-frequency region is the region of very low natural frequencies, where the rigidity of a weakly coupled system is enough to propagate the influence of boundary conditions throughout the whole structure. Let us designate the natural frequencies of the whole structure Ω_j, and the global normal modes of the structure $\mathbf{U}_j(\mathbf{r})$ ($j = 1, 2, \ldots, J$).

However, a region of high frequencies exists as well. In so far as the global resonances are impossible in the frequency region, it is clear that the vibration localizes within each substructure or group of substructures, but not within the whole structure. The cause of such localization is clear: the structure, being weakly coupled, does not possess the necessary global rigidity. The phenomenon of mode localization was first mentioned by Mandelstamm (1929). A detailed survey of mode localization phenomena in structures can be found in the review paper by Ibrahim (1987). From later papers it is also worth mentioning the papers by Cornwell and Bendiksen (1989) and Pierre and Cha (1989). Research by the mechanisists has shown that the phenomenon of mode localization occurs not only for disordered assemblies of weakly coupled subsystems, but even for periodic structures such as bladed-disk assemblies. Because of the localization of vibration within the substructures, we may use expansion (1), where the $\mathbf{u}_{nk}(\mathbf{r})$ are the normal modes of separate substructures.

The mathematical description of these arguments is as follows:

$$\mathbf{u}_c(\mathbf{r}, \omega) = \begin{cases} \sum_{j=1}^{\infty} \mathbf{U}_j(\mathbf{r}) Q_j(\omega), & 0 < |\omega| < \Omega \\ \mathbf{u}(\mathbf{r}, \omega), & |\omega| > \Omega. \end{cases} \tag{3}$$

Here $Q_j(\omega)$ is the spectrum of the generalized coordinate $Q_j(t)$, corresponding to the normal mode $\mathbf{U}_j(\mathbf{r})$. The parameter Ω plays the role of a boundary separating the two frequency regions mentioned. This parameter is hypothetical and cannot be denoted exactly. Moreover, as will be shown later, there is no necessity to know the exact value of Ω since this value will not be present in the final results.

In spite of the fact that formula (3) is no more than the designation of the spectrum $\mathbf{u}_c(\mathbf{r}, \omega)$ in different frequency regions, it is convenient to present $\mathbf{u}_c$ in the form of the sum of two functions and rewrite (1) in such a way that

$$\mathbf{u}_n(\mathbf{r}, t) = \sum_{k=1}^{\infty} \mathbf{u}_{nk}(\mathbf{r}) q_{nk}(t) + \sum_{j=1}^{J} \mathbf{U}_j(\mathbf{r}) Q_j(t) + \mathbf{u}(\mathbf{r}, t). \tag{4}$$

It is assumed that the spectra of the functions $Q_j(t)$ and $\mathbf{u}(\mathbf{r}, t)$, being localized in the low-frequency and high-frequency regions respectively, do not intersect. The representation (4) is twice overdetermined. Hence, to get the boundary problem for all unknown quantities in (4) it is necessary to impose two additional conditions.

Let us specify the type of the normal modes $\mathbf{u}_{nk}(\mathbf{r})$. Let us require that the normal modes vanish on the surface S_n, i.e. $\mathbf{u}_{nk}(\mathbf{r})$ are the nontrivial solutions of the following boundary-value problem

$$\begin{aligned} &\mathbf{r} \in V_n \quad \nabla \cdot [\mathbf{C} \cdot\cdot (\nabla \mathbf{u}_{nk})] + \rho \omega_{nk}^2 \mathbf{u}_{nk} = 0, \\ &\mathbf{r} \in S_n \quad \mathbf{u}_{nk} = 0. \end{aligned} \tag{5}$$

Here $\rho(\mathbf{r})$ is the density, $\mathbf{C}(\mathbf{r})$ is the tensor of elastic moduli, ω_{nk} is the kth eigenfrequency of the substructure V_n, the symbols $\cdot$ and $\cdot\cdot$ denote the operation of scalar and double scalar multiplication, and ∇ is the Hamiltonian operator. If all normal modes vanish on the surface S_n, the function $\mathbf{u}_c(\mathbf{r}, t)$ coincides with the veritable displacement $\mathbf{u}_n(\mathbf{r}, t)$ on the surfaces of all substructures. Aiming at obtaining the engineering theory for the integral description of the high-frequency vibration fields, it is natural to require extreme smoothness of the function $\mathbf{u}(\mathbf{r}, t)$ within the whole structure V. Coinciding with the veritable displacement on the surfaces of all substructures, this quantity may be called the body displacement or the carrier structure displacement in the high-frequency region. In the low-frequency region, the function $\mathbf{u}_c(\mathbf{r}, t) = \sum_{j=1}^{J} \mathbf{U}_j(\mathbf{r}) Q_j(t)$ represents the displacement of the intermediate structure, according to the terminology of Hale and Meirovitch (1980) and Meirovitch (1980). The various methods for dynamic simulation are considered in these papers.

The kinetic energy of the structure, considering expansion (4), allows the following representation:

$$\begin{aligned} T = \frac{1}{2} \sum_{n=1}^{N} \int_{V_n} \rho \dot{\mathbf{u}}_n \cdot \dot{\mathbf{u}}_n \, dV = \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{\infty} \dot{q}_{nk}^2 + \frac{1}{2} \sum_{j=1}^{J} \dot{Q}_j^2 + \frac{1}{2} \int_V \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dV \\ + \sum_{n=1}^{N} \sum_{k=1}^{\infty} \sum_{j=1}^{J} \dot{q}_{nk} \dot{Q}_j \int_{V_n} \rho \mathbf{u}_{nk} \cdot \mathbf{U}_j \, dV + \sum_{n=1}^{N} \sum_{k=1}^{\infty} \dot{q}_{nk} \int_{V_n} \rho \mathbf{u}_{nk} \cdot \dot{\mathbf{u}} \, dV. \end{aligned} \tag{6}$$

The property of normal modes to be orthonormal within each substructure

$$\int_{V_n} \rho \mathbf{u}_{nk} \cdot \mathbf{u}_{ns} \, dV = \delta_{ks}; \quad \int_{V_n} (\nabla \mathbf{u}_{nk}) \cdot\cdot \mathbf{C} \cdot\cdot (\nabla \mathbf{u}_{ns}) \, dV = \omega_{nk}^2 \delta_{ks},$$

and the same property of the global modes

$$\int_V \rho \mathbf{U}_j \cdot \mathbf{U}_s \, dV = \delta_{js}; \quad \int_V (\nabla \mathbf{U}_j) \cdot\cdot \mathbf{C} \cdot\cdot (\nabla \mathbf{U}_s) \, dV = \Omega_j^2 \delta_{js}.$$

are taken into account. The cross summand $\int_{V_n} \rho\dot{\mathbf{u}} \cdot \sum_{j=1}^{J} \mathbf{U}_j \dot{Q}_j \, dV$ is omitted because $\mathbf{u}$ and $\sum_{j=1}^{J} \mathbf{U}_j Q_j$ denote the same function $\mathbf{u}_c$, in different frequency regions. In view of the extreme smoothness of $\mathbf{u}$ and essential heterogeneity of the structure, the next estimate would appear to be correct:

$$\int_V \rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dV = \sum_{n=1}^{N} (\dot{\mathbf{u}})_n \cdot (\dot{\mathbf{u}})_n \int_{V_n} \rho \, dV = \sum_{n=1}^{N} \langle\rho\rangle (\dot{\mathbf{u}})_n \cdot (\dot{\mathbf{u}})_n V_n = \int_V \langle\rho\rangle \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dV, \tag{7}$$

where $\langle\rho\rangle = (1/V)\int_V \rho \, dV$ is the average density of the structure. The latter equality in (7) signifies the standard transition from the Riemann–Stieltjes sum, to the corresponding integral. The transition is quite permissible for large N. In the last summand in formula (6), $\rho(\mathbf{r})$ and $\mathbf{u}_{nk}(\mathbf{r})$ are highly oscillating functions of $\mathbf{r}$, while $\mathbf{u}(\mathbf{r}, t)$ is an extremely smooth function with respect to $\mathbf{r}$, which may be placed beyond the integral. If we introduce the average displacement of the centre of mass of the substructure V_n when it moves according to mode k,

$$\langle \mathbf{u}_{nk} \rangle = \frac{1}{\langle\rho\rangle_n V_n} \int_{V_n} \rho \mathbf{u}_{nk} \, dV,$$

expression (6) can be rewritten in the following form:

$$T = \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{r} \dot{q}_{nk}^2 + \frac{1}{2} \sum_{j=1}^{J} \dot{Q}_j^2 + \frac{1}{2} \int_V \langle\rho\rangle \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dV + \sum_{n=1}^{N} \sum_{k=1}^{r} \sum_{j=1}^{J} \alpha_{nkj} \dot{q}_{nk} \dot{Q}_j + \sum_{n=1}^{N} \sum_{k=1}^{r} \langle\rho\rangle \langle \mathbf{u}_{nk} \rangle \cdot \dot{\mathbf{u}} V_n \dot{q}_{nk}. \tag{8}$$

Here $\alpha_{nkj} = \int_{V_n} \rho \mathbf{u}_{nk} \cdot \mathbf{U}_j \, dV$ is an unknown coefficient of expansion of the global mode $\mathbf{U}_j$ in terms of the normal modes $\mathbf{u}_{nk}$ of substructures, i.e.

$$\mathbf{r} \in V_n \quad \mathbf{U}_j(\mathbf{r}) = \sum_{k=1}^{r} \alpha_{nkj} \mathbf{u}_{nk}(\mathbf{r}). \tag{9}$$

The representation of the potential energy is obtained in a similar way (Belyaev and Palmov, 1986):

$$\Pi = \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{r} \omega_{nk}^2 q_{nk}^2 + \frac{1}{2} \sum_{j=1}^{J} \Omega_j^2 Q_j^2 + \frac{1}{2} \int_V (\nabla\mathbf{u}) \cdot\cdot \langle \mathbf{C} \rangle \cdot\cdot (\nabla\mathbf{u}) \, dV + \sum_{n=1}^{N} \sum_{k=1}^{r} \sum_{j=1}^{J} \alpha_{nkj} \omega_{nk}^2 q_{nk} Q_j. \tag{10}$$

Here $\langle \mathbf{C} \rangle$ is a certain mean elastic moduli tensor over the structure.

The work of the external loads is

$$W = \sum_{n=1}^{N} \sum_{k=1}^{r} p_{nk} q_{nk} + \sum_{j=1}^{J} P_j Q_j + \int_V \mathbf{g} \cdot \mathbf{u} \, dV + \int_S \mathbf{f} \cdot \mathbf{u} \, dS, \tag{11}$$

where the generalized forces p_{nk}, P_j are

$$p_{nk} = \int_V \mathbf{g} \cdot \mathbf{u}_{nk} \, dV; \quad P_j = \int_V \mathbf{g} \cdot \mathbf{U}_j \, dV + \int_S \mathbf{f} \cdot \mathbf{U}_j \, dS. \tag{12}$$

Using the Hamiltonian variational principle, one obtains the following boundary-value problem:

$$\mathbf{r}\in V \quad \ddot{Q}_j+\Omega_j^2 Q_j+\sum_{k=1}^{\infty}(\ddot{q}_{nk}+\omega_{nk}^2 q_{nk})\alpha_{nkj}=P_j, \quad j=1,2,\ldots,J \tag{13}$$

$$\nabla\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]-\langle\rho\rangle\left[\ddot{\mathbf{u}}+\sum_{k=1}^{\infty}\langle\mathbf{u}_{nk}\rangle\ddot{q}_{nk}\right]+\mathbf{g}=0. \tag{14}$$

$$\mathbf{r}\in V_n \quad \ddot{q}_{nk}+\omega_{nk}^2 q_{nk}+\sum_{j=1}^{J}\alpha_{nkj}(\ddot{Q}_j+\omega_{nk}^2 Q_j)=p_{nk}-\langle\rho\rangle\langle\mathbf{u}_{nk}\rangle\cdot\ddot{\mathbf{u}}V_n;$$

$$k=1,2,\ldots,\infty, \quad n=1,2,\ldots,N, \tag{15}$$

$$\mathbf{r}\in S \quad \mathbf{N}\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]=\mathbf{f}, \tag{16}$$

where $\mathbf{N}$ is the unit vector of the external normal to the surface S.

It is now necessary to remember that we have put an additional condition on the high-frequency vibration (i.e. $\mathbf{u}$), but not on the low-frequency vibration. Nevertheless, if $\mathbf{U}_j(\mathbf{r})$ is the global mode, indeed, the equation for generalized coordinate $Q_j(t)$ must be the following:

$$\ddot{Q}_j+\Omega_j^2 Q_j=P_j. \tag{17}$$

Taking into account (13) the latter equation is equivalent to the conditions

$$\sum_{k=1}^{\infty}\alpha_{nkj}(\ddot{q}_{nk}+\omega_{nk}^2 q_{nk})=0. \tag{18}$$

Relations (18) are actually a system of equations to find α_{nkj} or $\mathbf{U}_j(\mathbf{r})$ with the help of (9). Formula (9) is a classical formula for an arbitrary function to be represented as a series of orthonormal functions $\mathbf{u}_{nk}(\mathbf{r})$, within each substructure V_n. This system of functions is complete and in some way, a natural one for the substructure in question. Nevertheless, the search for the solution in the form (9) is the most unsuccessful and unnatural one to work with computationally. In other words, this approach is not constructive (Hale and Meirovitch, 1980). Really, first: the definition of the substructural normal modes as a result of solving the eigenvalue problem, is a serious task and is in no way simpler than solving the problem of structural global modes. Second: the system of the substructure eigenfunctions obtained is so complicated that it is impossible to work with, in practice. Third: the boundary-value problem for the substructure is not unique, i.e. any boundary condition would be doubtful (Meirovitch, 1980).

At the same time, as noted by Hale and Meirovitch (1980) and Meirovitch (1980), for finding the first several global modes of the complicated mechanical structure, the limited set of suitable spatial functions (so called "admissible functions") is sufficient. Moreover, these functions in the energy space of substructures are not required to satisfy any boundary conditions in the internal boundaries of the substructure. Hence, they may be chosen according to any suitable principle, for instance, convenience of interpretation. To create the theory in question, it was convenient to accept the condition that the normal modes vanish at the boundaries of the substructures. This argument may be considered as a type of foundation for choosing the normal modes set $\mathbf{u}_{nk}(\mathbf{r})$, as a nontrivial solution of the boundary-value problem (5). In any case, it is excessive to try to obtain the global normal modes using the expansion (9). The global normal modes $\mathbf{U}_j(\mathbf{r})$ and global eigenfrequencies Ω_j may be obtained using methods of computational structural mechanics; for instance BEM, FEM, substructure synthesis methods, etc. (Noor and Atluri, 1987; Meirovitch, 1980).

Let us assume that the external loads allow the following spectral decomposition:

$$\mathbf{g}(\mathbf{r},t)=\int_{-\infty}^{+\infty}\mathbf{g}(\mathbf{r},\omega)\,e^{i\omega t}\,d\omega;\quad \mathbf{f}(\mathbf{r},t)=\int_{-\infty}^{+\infty}\mathbf{f}(\mathbf{r},\omega)\,e^{i\omega t}\,d\omega. \tag{19}$$

Then the boundary-value problem (14)–(17) for the spectra $\mathbf{u}(\mathbf{r},\omega)$, $Q_j(\omega)$, $q_{nk}(t)$ has the form

$$\mathbf{r}\in V\quad (-\omega^2+2i\Psi_j\Omega_j\omega+\Omega_j^2)Q_j=P_j \tag{20}$$

$$\nabla\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]+\langle\rho\rangle\omega^2\left(\mathbf{u}+\sum_{k=1}^{\infty}\langle\mathbf{u}_{nk}\rangle q_{nk}\right)+\mathbf{g}=0, \tag{21}$$

$$\mathbf{r}\in V_n\quad (-\omega^2+2i\psi_{nk}\omega_{nk}\omega+\omega_{nk}^2)\left(q_{nk}+\sum_{j=1}^{J}\alpha_{nkj}Q_j\right)=p_{nk}+\omega^2\langle\rho\rangle\langle\mathbf{u}_{nk}\rangle\cdot\mathbf{u}V_n \tag{22}$$

$$\mathbf{r}\in S\quad \mathbf{N}\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]=\mathbf{f}. \tag{23}$$

Here ψ_{nk}, Ψ_j are the dimensionless coefficients of damping of the corresponding normal modes of the substructures and the whole structure. Obtaining q_{nk} from (20), (22) and substituting it into (21), we get the following differential equation for high-frequency vibration:

$$\nabla\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]+\omega^2\mathbf{A}(\omega)\cdot\mathbf{u}+\mathbf{g}_e=0. \tag{24}$$

Here

$$\mathbf{g}_e=\mathbf{g}+\langle\rho\rangle\omega^2\sum_{k=1}^{\infty}\frac{p_{nk}\langle\mathbf{u}_{nk}\rangle}{-\omega^2+2i\psi_{nk}\omega_{nk}\omega+\omega^2}-\langle\rho\rangle\omega^2\sum_{j=1}^{J}\frac{P_j\alpha_{nkj}\langle\mathbf{u}_{nk}\rangle}{-\omega^2+2i\Psi_j\Omega_j\omega+\Omega_j^2}, \tag{25}$$

$$\mathbf{A}(\omega)=\langle\rho\rangle\left[\mathbf{E}+\langle\rho\rangle\sum_{k=1}^{\infty}\frac{\langle\mathbf{u}_{nk}\rangle\langle\mathbf{u}_{nk}\rangle V_n}{-\omega^2+2i\psi_{nk}\omega_{nk}\omega+\omega_{nk}^2}\right], \tag{26}$$

where $\mathbf{g}_e(\omega)$ is the spectrum of the effective load per unit volume, and $\mathbf{A}(\omega)$ is the tensor of mass inertia of the structure. We accept the hypothesis of this tensor isotropy (Belyaev and Palmov, 1986), because actual mechanical structures have such a complicated composition that it is impossible to find the anisotropic axis of the substructures' spectral characteristics. In conclusion, the boundary-value problem for the integral description of the structure's vibration is

$$\mathbf{u}_e(\mathbf{r},\omega)=\sum_{k=1}^{\infty}\mathbf{U}_j(\mathbf{r})Q_j(\omega)+\mathbf{u}(\mathbf{r},\omega) \tag{27}$$

$$\mathbf{r}\in V\quad (-\omega^2+2i\Psi_j\Omega_j\omega+\Omega_j^2)Q_j=P_j\quad j=1,2,\ldots,J \tag{28}$$

$$\nabla\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]+\omega^2A(\omega)\mathbf{u}+\mathbf{g}_e=0, \tag{29}$$

$$\mathbf{r}\in S\quad \mathbf{N}\cdot[\langle\mathbf{C}\rangle\cdot\cdot(\nabla\mathbf{u})]=\mathbf{f} \tag{30}$$

where

$$A(\omega)=\tfrac{1}{3}\mathbf{A}(\omega)\cdot\cdot\mathbf{E}=\langle\rho\rangle\left[1+\langle\rho\rangle\sum_{k=1}^{\infty}\frac{\omega^2\langle\mathbf{u}_{nk}\rangle\cdot\langle\mathbf{u}_{nk}\rangle V_n}{3(-\omega^2+2i\psi_{nk}\omega_{nk}\omega+\omega_{nk}^2)}\right]. \tag{31}$$

$A(\omega)$ is formed by an infinite number of resonance curves corresponding to the vibration of a single-degree-of-freedom system. Let us place the components of eqn (31) in ascending order of natural frequencies. The width of each resonance curve is $2\psi_{nk}\omega_{nk}$ at a level $1/\sqrt{2}$ from the resonance value. If the intervals between adjacent natural frequencies are less than or approximately coincide with the width of the resonance curves

$$\Delta\omega_{nk} = \omega_{nk+1} - \omega_{nk} \leqslant \psi_{nk}\omega_{nk} + \psi_{nk+1}\omega_{nk+1}. \tag{32}$$

or because $\omega_{nk+1} \approx \omega_{nk}$

$$\frac{\Delta\omega_{nk}}{\omega_{nk}} \leqslant 2\psi_{nk}. \tag{33}$$

the resonance curves in (31) merge, forming a smooth frequency function. In this case, eqn (31) can be replaced by the integral over the high-frequency region

$$A(\omega) = \langle\rho\rangle\left[1 + \omega^2 \int_{\Omega}^{\infty} \frac{\Phi(\alpha)\,d\alpha}{-\omega^2 + 2i\psi\alpha\omega + \alpha^2}\right]. \tag{34}$$

where a locally smooth function of the natural frequency distribution $\Phi(\alpha)$ is introduced, so that

$$\Phi(\omega_{nk})\Delta\omega_{nk} = \tfrac{1}{3}\langle\rho\rangle\langle\mathbf{u}_{nk}\rangle\cdot\langle\mathbf{u}_{nk}\rangle V_n. \tag{35}$$

We have assumed that the spectral properties are identical within the whole structure, since the spectral characteristics of the structure can only be obtained as a result of certain experiments of excitation of narrow-band or single frequency vibration of the whole structure. Instead of (34), we can now write

$$A(\omega) = \langle\rho\rangle[\delta(\omega) - i\kappa(\omega)]^2 \tag{36}$$

where $\delta(\omega)$ and $\kappa(\omega)$ are nondimensional frequency-dependent parameters. As shown by Belyaev and Palmov (1986), the parameter $\delta(\omega)$ allows the following estimation: $\delta(\omega) \approx 1$. We obtain, after comparing (34) and (36),

$$\kappa(\omega) = \omega^3 \int_{\Omega}^{\infty} \frac{\psi\alpha\Phi(\alpha)\,d\alpha}{(\alpha^2 - \omega^2)^2 + 4\psi^2\alpha^2\omega^2}. \tag{37}$$

Assuming a small value of damping ($\psi \ll 1$) and local smoothness of $\Phi(\alpha)$, integral (37) can be estimated by methods of the random vibration theory (Bolotin, 1969), i.e.

$$\kappa(\omega) = \tfrac{1}{2}\pi\omega\Phi(\omega). \tag{38}$$

From this last formula we can see that $\kappa(\omega)$ and consequently, the value of absorption of high-frequency vibration, does not depend on the damping ψ. Actually, this coefficient is absent in eqn (38). The value of absorption is determined, first of all, by the distribution function $\Phi(\alpha)$. It is natural because the internal degrees of freedom act as a set of dynamical absorbers, with respect to the carrier structure. If the damping is not small, the resonance curves corresponding to the internal degrees of freedom merge. In this case, the considerable spatial absorption of vibration for the whole high-frequency region is guaranteed. This effect was mentioned by Der Kiureghian and Igusa (1987) by way of a numerical example and parametric study.

3. AN EXAMPLE: THE LONGITUDINAL VIBRATION OF AN EXTENDED STRUCTURE

The solution of the boundary-value problem is demonstrated in the following example. Let us consider an extended complex mechanical structure, where the excitation of the structure is such that longitudinal waves predominate. In this case the structure can be represented as a rod. Let the rod's length be l, the cross-section $x = l$ be under a force $F(t)$ and the cross-section $x = 0$ be free. The global natural frequencies Ω_j and normal modes $U_j(x)$ are assumed to be calculable using a method of structural mechanics (Meirovitch, 1980). The volume load is absent ($g = 0$), hence $g_e = 0$. The boundary-value problem (27)–(30) can thus be written as

$$u_e(x,\omega) = \sum_{j=1}^{\infty} U_j(x)Q_j(\omega) + u(x,\omega) \tag{39}$$

$$Q_j = \frac{U_j(l)F(\omega)}{-\omega^2 + 2i\psi_j\Omega_j\omega + \Omega_j^2} \tag{40}$$

$$C\frac{d^2u}{dx^2} + \omega^2 M(\delta - i\kappa)^2 u = 0 \tag{41}$$

$$x = 0 \quad \frac{du}{dx} = 0; \quad x = l \quad C\frac{du}{dx} = F(\omega), \tag{42}$$

where C is the longitudinal rigidity and M is the mass of the length unit. The solution of the boundary problem (41), (42) is as follows:

$$u(x,\omega) = -\frac{F(\omega)}{C\lambda} \cdot \frac{\cos \lambda x}{\sin \lambda l} \tag{43}$$

where $\lambda = (\omega/a)(\delta - i\kappa)$ is the wave number and $a = \sqrt{C/M} = \sqrt{E/\rho}$ is the velocity of the energetic centre of the propagating disturbance.

Taking into account formula (2), the field of the structure's acceleration looks like:

$$\ddot{u}_e(x,t) = \int_{-\infty}^{+\infty} -\omega^2 F(\omega)\left[\sum_{j=1}^{J} \frac{U_j(l)U_j(x)}{-\omega^2 + 2i\Psi_j\Omega_j\omega + \Omega_j^2} - \frac{a}{C\omega(\delta - i\kappa)} \cdot \frac{\cos\frac{\omega}{a}(\delta - i\kappa)x}{\sin\frac{\omega}{a}(\delta - i\kappa)l}\right] e^{i\omega t}\, d\omega. \tag{44}$$

Let us assume that the external load is a stationary random function having spectral density $S_F(\omega)$. The spectral density of the acceleration may be calculated according to standard methods of the random vibration theory

$$S_{\ddot{u}}(x,\omega) = \omega^4 S_F(\omega)\left|\sum_{j=1}^{J} \frac{U_j(l)U_j(x)}{-\omega^2 + 2i\Psi_j Q_j\omega + \Omega_j^2} - \frac{a}{C\omega(\delta - i\kappa)} \cdot \frac{\cos\frac{\omega}{a}(\delta - i\kappa)x}{\sin\frac{\omega}{a}(\delta - i\kappa)l}\right|^2. \tag{45}$$

After separating the real and imaginary parts, the result (45) can be rewritten as

$$S_a(\omega) = \omega^4 S_F(\omega)\left\{\left|\sum_{j=1}^{J}\frac{U_j(l)U_j(x)}{-\omega^2+2i\Psi_j\Omega_j\omega+\Omega_j^2}\right|^2\right.$$

$$-\frac{a}{C\omega}\left[\frac{\cos\frac{\omega}{a}(\delta+i\kappa)x}{(\delta+i\kappa)\sin\frac{\omega}{a}(\delta+i\kappa)l}\sum_{j=1}^{J}\frac{U_j(l)U_j(x)}{-\omega^2+2i\Psi_j\Omega_j\omega+\Omega_j^2}\right.$$

$$\left.+\frac{\cos\frac{\omega}{a}(\delta-i\kappa)x}{(\delta-i\kappa)\sin\frac{\omega}{a}(\delta-i\kappa)l}\sum_{j=1}^{J}\frac{U_j(l)U_j(x)}{-\omega^2-2i\Psi_j\Omega_j\omega+\Omega_j^2}\right]$$

$$\left.+\frac{a^2}{C^2\omega^2(\delta^2+\kappa^2)}\,\frac{\operatorname{ch}\frac{2\omega\kappa x}{a}+\cos\frac{2\omega\delta x}{a}}{\operatorname{ch}\frac{2\omega\kappa l}{a}-\cos\frac{2\omega\delta l}{a}}\right\}. \tag{46}$$

The latter formula is inordinately unwieldy, but it may be possible to simplify. For this purpose, it is necessary to notice that even for the smooth function $S_F(\omega)$, the spectral density of the acceleration $S_a(\omega)$ is a highly oscillating function of frequency in the high-frequency region because of the several trigonometrical functions. It is reasonable to average the right part of the dependence (46) within some frequency range using, for instance, the averaging method suggested by Palmov (1976) for a viscoelastic rod.

Let us consider the last term in (46). In the high-frequency region the hyperbolic functions are large and the additions from the trigonometrical functions are negligible, and only lead to the "unneeded" complication of smooth dependence. To average over the period of each trigonometrical function, we replace the term

$$\frac{\operatorname{ch}\frac{2\omega\kappa x}{a}+\cos\frac{2\omega\delta x}{a}}{\operatorname{ch}\frac{2\omega\kappa l}{a}-\cos\frac{2\omega\delta l}{a}}$$

by the following integral:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{dz_1}{\operatorname{ch}\frac{2\omega\kappa l}{a}-\cos z_1}\cdot\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(\operatorname{ch}\frac{2\omega\kappa x}{a}+\cos z_2\right)dz_2. \tag{47}$$

The variables $z_1 = 2\omega\delta l/a$ and $z_2 = 2\omega\delta x/a$ may be considered as independent because the values of x and l are generally incommensurable. The result of integration is the following (Gradshtein and Ryzhik, 1980):

$$\frac{\operatorname{ch}\frac{2\omega\kappa x}{a}}{\operatorname{sh}\frac{2\omega\kappa l}{a}}.$$

The terms in the square bracket in (46) vanish after analogous averaging. The terms

corresponding to the global vibration of the structure are not averaged, since they represent the smooth frequency dependence in the high-frequency region. Hence, the dependence (46), smoothed within several oscillations, may be written as

$$S_u(\omega) = \omega^4 S_F(\omega)\left\{\sum_{j=1}^{J} \frac{U_j^2(l)U_j^2(x)}{(-\omega^2+\Omega_j^2)^2+4\Psi_j^2\Omega_j^2\omega^2} + \frac{a^2}{C^2\omega^2(\delta^2+\kappa^2)} \frac{\operatorname{ch}\dfrac{2\omega\kappa x}{a}}{\operatorname{sh}\dfrac{2\omega\kappa l}{a}}\right\}, \quad (48)$$

where it has been taken into account that the first resonance curves of the structure are distant from each other.

The result (48) may be obtained just from (45) if one remembers that there are low-frequency and high-frequency regions [see (3)]. After averaging in the high-frequency region, formula (45) can be shown to be

$$S_u(\omega) = \omega^4 S_F(\omega)\begin{cases} \left|\sum_{j=1}^{J} \dfrac{U_j(l)U_j(x)}{-\omega^2+2i\Psi_j\Omega_j\omega+\Omega_j^2}\right|^2, & \omega < \Omega \\ \dfrac{a^2}{C^2\omega^2(\delta^2+\kappa^2)} \dfrac{\operatorname{ch}\dfrac{2\omega\kappa x}{a}}{\operatorname{sh}\dfrac{2\omega\kappa l}{a}}, & \omega > \Omega. \end{cases} \quad (49)$$

It is now simple to obtain the result (48) by joining the formulae in (49) and taking into account the absence of the mutual correlation of the frequency regions.

The more complicated way chosen by us, allows for the solution of the problem without using the boundary frequency Ω. Hence, there is no need to get the precise value of Ω. The important thing for us is the actual existence of such a parameter, which may be implicitly used for creating the theory and also in simplifying practical problems.

4. SOME NUMERICAL RESULTS

Let us suppose that some complex mechanical system may be represented as a tube of the length $l = 10$ m, with diameter $D = 2R = 1$ m. Let the longitudinal rigidity of the system be concentrated in the tube's shell of thickness $h = 2 \times 10^{-3}$ m. Let the Young's modulus of the material be $E = 7 \times 10^{10}$ N m^{-2}. Let us suppose also, that all the contents of the system are fixed to the tube's shell. Let the average density of the system be $\langle\rho\rangle = 10^3$ kg m^{-3}. For the sake of simplicity, we assume that the first global modes and natural frequencies coincide with the corresponding ones of a continuous homogeneous rod, having longitudinal rigidity $C = 2\pi hRE = 4.40 \times 10^8$ N and mass per unit length $M = \pi R^2\rho = 7.85 \times 10^2$ kg m^{-1}. The eigenfrequencies and the orthonormal modes of the longitudinal vibration of a free rod are

$$\Omega_j = j\frac{\pi}{l}\sqrt{\frac{C}{M}}, \quad U_j(x) = \sqrt{\frac{2}{Ml}}\cos j\pi\frac{x}{l}. \quad (50)$$

Let us suppose that only two first global resonances are possible in this system. The simplest calculation gives the following values of global eigenfrequencies: $f_1 = (1/2\pi)\Omega_1 = 37.4$ cps, $f_2 = 2f_1 = 74.8$ cps.

The dependence of the square of the absolute value of the transfer function

$$|H|^2 = \frac{S_u(x,\omega)}{S_F(\omega)},$$

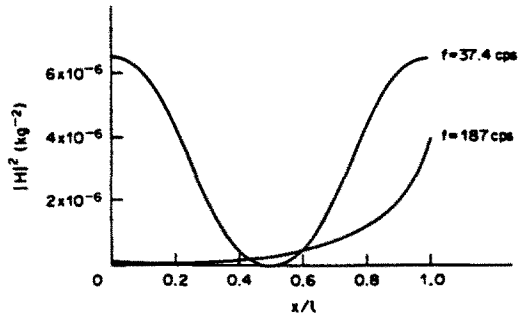

Fig. 1. Square of the absolute value of the transfer function as a function of x/l for various frequencies.

with respect to x and cyclic frequency $f = (1/2\pi)\omega$ is represented in Figs 1 and 2. The following values of damping are assumed: $\psi = 0.05$, $\kappa = 0.2$ (Palmov, 1976). The dependences of $|H|^2$ on x/l for $f = f_1$ (low-frequency region) and for $f = 5f_1$ (high-frequency region) are presented in Fig. 1. The explicit difference is evident. In the low-frequency region the natural modes are easily observed. The rod's ends vibrate with the same amplitude, in spite of the fact that only one end is excited. The essential reduction of the level of vibration is typical for the high-frequency region. This reduction is an obvious result of mode localization. This effect occurs even in solid physics, e.g. the metallic conductivity is reduced because of localization of electron eigenstates (Anderson, 1958).

Let us now have a look at Fig. 2, where the dependence of $|H|^2$ on frequency for the cross-section $x = 0.9l$ is shown. The classical resonance curves are far from each other in the low-frequency region whereas such explicit resonance curves are impossible for a high-frequency vibration. In the high-frequency region the mechanical system has the amplitude–frequency characteristics which are typical of systems with a continuous spectrum of eigenfrequencies. For instance, infinite bodies have such spectra. In order to better observe this similarity, let us consider the same rod, but with cross-section $x = 0$ fixed. Instead of (48), (50) we now have

$$S_u(\omega) = \omega^4 S_F(\omega)\left\{\sum_{j=1}^{J} \frac{U_j^2(l)U_j^2(x)}{(-\omega^2+\Omega_j^2)^2+4\Psi_j^2\Omega_j^2\omega^2} + \frac{a^2}{C^2\omega^2(\delta^2+\kappa^2)} \cdot \frac{\operatorname{sh}\dfrac{2\omega\kappa x}{a}}{\operatorname{sh}\dfrac{2\omega\kappa l}{a}}\right\},$$

$$\Omega_j(j-\tfrac{1}{2})\frac{\pi}{l}\sqrt{\frac{C}{M}}, \quad U_j(x) = \sqrt{\frac{2}{Ml}}\sin(j-\tfrac{1}{2})\pi\frac{x}{l}. \tag{51}$$

The dependences of $|H|^2$ on x and f are presented in Figs 3 and 4. Comparison of transfer functions demonstrates the essential change in the low-frequency region. In the high-frequency region the shapes of the curves practically coincide; some insignificant changes are observed only near the end $x = 0$, but the values of $|H|^2$ are negligible there. Hence, the field of high-frequency vibration is not actually sensitive to a boundary condition at the unloaded end of the rod.

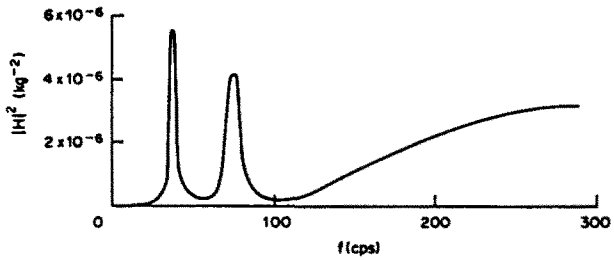

Fig. 2. Square of the absolute value of the transfer function as a function of frequency.

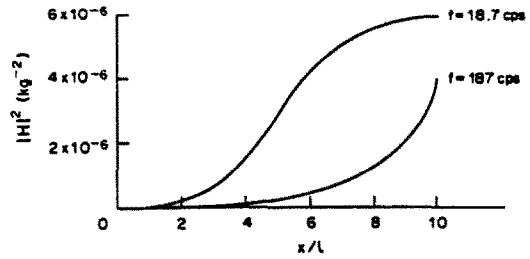

Fig. 3. Square of the absolute value of the transfer function as a function of x/l for various frequencies.

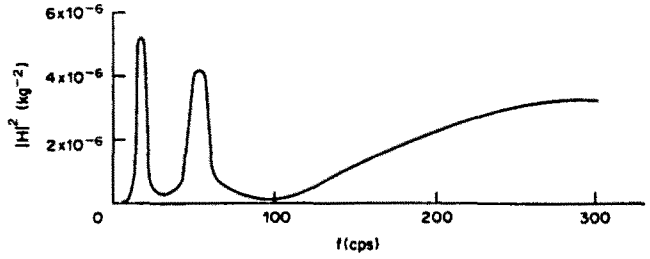

Fig. 4. Square of the absolute value of the transfer function as a function of frequency.

We may now formulate the main distinction between low- and high-frequency vibrations. In the low-frequency region, the global natural frequencies of the structure form the discrete spectrum, which is very sensitive to the boundary conditions. For high-frequency vibration, a complex mechanical structure appears to be an infinite body. The continuous spectrum of eigenfrequencies and insensitivity to the boundary conditions on the remote boundary surface confirm this conclusion. In other words: the low-frequency vibration envelops the whole, complex mechanical structure; the high-frequency vibration propagates from the source of excitation.

5. CONCLUSIONS

An integral description of the vibration of complex mechanical structures is proposed. It was shown that there are two frequency regions with essentially different properties. In the low-frequency region, the vibration envelops the whole structure and for dynamic simulation, it is reasonable to use common methods. These are actually integral ones. A very specific region of the high-frequency vibration was examined. In this region a structure acts as a mechanical system with a continuous spectrum of eigenfrequencies. The internal degrees of freedom of secondary structures correspond to a set of dynamic absorbers with respect to primary structure, thus providing considerable spatial absorption of the high-frequency vibration. The secondary structures may be damaged, since they absorb energy on their own eigenfrequencies. This is the main reason for paying attention to the high-frequency vibration of complex structures.

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